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The Wild Goose Chase Problem

Andrew Gard

Abstract. The classical pursuit problem considers the path traced by a point in space as it charges directly toward a moving target. But what if the target has more lofty goals than mere escape? To what degree can it control the path taken by its pursuer? We prove that the pursuer can be led to any point in $\mathbb{R}^n$ without being allowed to close more than an arbitrarily small distance en route.

1. INTRODUCTION. Let $\boldsymbol{p}(t) : \mathbb{R}^+ \to \mathbb{R}^n$ represent a path of *pure pursuit* taken by a point as it chases after a moving target $\boldsymbol{m}(t) : \mathbb{R}^+ \to \mathbb{R}^n$ in the most naive fashion possible:

$$\frac{d\boldsymbol{p}}{dt} = v\frac{\boldsymbol{m}(t) - \boldsymbol{p}(t)}{\|\boldsymbol{m}(t) - \boldsymbol{p}(t)\|}. \tag{1}$$

That is, the pursuer always aims directly at its target. Here $v = |\boldsymbol{p}'|$ is taken to be constant, reflecting the natural assumption that the pursuer always chooses to move as fast as possible.

This problem was first analyzed mathematically in 1732 by the Pierre Bouguer, the so-called father of naval architecture [1]. For this 18th century Frenchman, the target was a merchant vessel heading along a linear course $\boldsymbol{m}(t) = (R_0, t)$ in the plane, while the pursuer was a pirate beginning at the origin. The situation is illustrated in Figure 1.

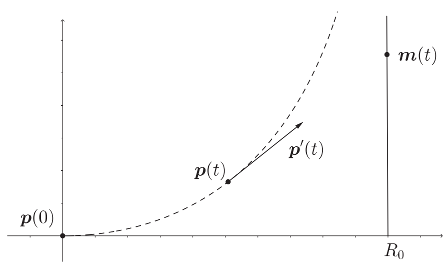

Figure 1. A pirate at work.

By eliminating the time parameter and applying some modest manipulation, the path of the pirate ship can be computed by hand. For instance, when $v = 1$,

$$y = \frac{1}{2}R_0\left[\frac{1}{2}\left(1 - \frac{x}{R_0}\right)^2 - \ln\left(1 - \frac{x}{R_0}\right)\right] - \frac{1}{4}R_0.$$

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Interestingly, under the assumption made here that the pursuer and target move at identical speeds, the distance between them decreases asymptotically to $R_0/2$. A lovely analysis of this problem can be found in Paul Nahin's gem of popular mathematics, *Chases and Escapes* [6].

The classical pursuit equation remains fertile ground for exploration. Undergraduate researchers need only substitute a new flight path $\boldsymbol{m}(t)$ into equation (1), convince Matlab to spit out some pretty plots, dig into some obvious questions (Arc length? Time to capture? Asymptotic distance?), and they are practically ready for their poster sessions.

For instance, imagine a gull diving after a thrown morsel of food. The parameters in such a situation are the starting position of the gull and the angle and initial velocity of the throw. Under what circumstances can the gull make a midair catch? A typical pursuit curve is shown in Figure 2.

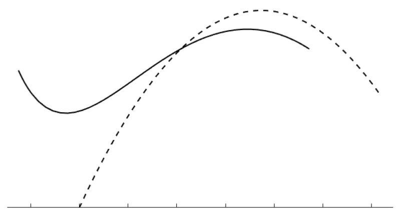

Figure 2. A bird after its dinner.

2. THE INTRINSIC PERSPECTIVE. Still, there is something unsatisfying about analyzing pursuit questions using fixed extrinsic coordinates. While it may be conceivable that a ship at sea would continue on an absolute course (straight or otherwise) despite a pirate bearing down from its flank, it seems more natural to think that it would instead choose its bearing *relative* to the pursuer. Wouldn't any sensible merchant turn to head directly away from those murderous buccaneers regardless of what the compass might read?

A more natural perspective on the situation is illustrated in Figure 3.

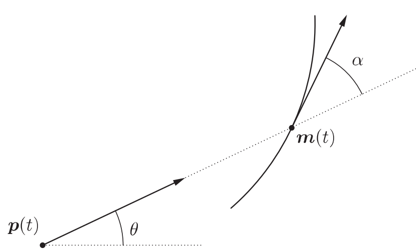

Figure 3. The intrinsic perspective.

Setting $|\boldsymbol{p}'(t)| = |\boldsymbol{m}'(t)| = 1$, we can obtain a complete description of the pursuit problem in terms of the relative bearing $\alpha(t)$ of the merchant, the curvature $\kappa(t)$ of the path of the pursuer, and the distance $R(t)$ between them. Note that this includes the *natural* or *intrinsic* description of the trace of $\boldsymbol{p}$.

A brief calculation translates the classical pursuit equation into these new variables. Since $R(t)\boldsymbol{p}'(t) = \boldsymbol{m}(t) - \boldsymbol{p}(t)$, by equation (1) we have

$$\boldsymbol{m}'(t) = \boldsymbol{p}'(t) + R'(t)\boldsymbol{p}'(t) + R(t)\boldsymbol{p}''(t),$$

which offers a decomposition in terms of the normal and tangential components of $\boldsymbol{p}'$:

$$\boldsymbol{m}'(t) = \left[1 + R'(t)\right]\boldsymbol{T_p} + \left[R(t)\kappa(t)\right]\boldsymbol{N_p}.$$

Noting that $\langle \boldsymbol{m}'(t), \boldsymbol{T_p}\rangle = \cos\alpha(t)$ and $\langle \boldsymbol{m}'(t), \boldsymbol{N_p}\rangle = \sin\alpha(t)$, we obtain the *intrinsic form of the pursuit equation*:

$$\frac{dR}{dt} = -1 + \cos\alpha(t), \qquad R\kappa = \sin\alpha(t). \tag{2}$$

The first of these gives a *rate of closing* on the target while the second (after division by R) gives a *rate of turning*. For a naval commander, for instance, this is more pertinent information than any representation in extrinsic coordinates.

There is no information lost in such a representation. Setting $\boldsymbol{p}(0) = (0, 0)$ and $\boldsymbol{m}(0) = (R_0, 0)$ as before, the original description of the pursuit curve $\boldsymbol{p}(t) = (x(t), y(t))$ is recovered using the fundamental theorem of plane curves [2]:

$$x(t) = \int_0^t \cos\theta(\xi)\,d\xi, \qquad y(t) = \int_0^t \sin\theta(\xi)\,d\xi,$$

where $\theta(\xi) = \int_0^\xi \kappa(\tau)\,d\tau$.

In practice these integrals are an ugly mess and should only be approached numerically. However, the test case $\alpha(t) = \alpha_0$, in which the target veers off at a constant angle, is perfectly tractable. Solving the intrinsic pursuit equations yields

$$R(t) = (-1 + \cos\alpha_0)t + R_0, \qquad \kappa(t) = \frac{\sin\alpha_0}{(-1 + \cos\alpha_0)t + R_0}.$$

Readers may recognize this curvature function κ: it defines a *logarithmic spiral*, a curve with long history in the literature of pursuit and evasion. In the n-bug problem, for instance, in which a number of identical bugs begin at the corners of a regular polygon and chase directly at one another, the resulting arcs turn out to be logarithmic spirals. The $n = 4$ case is illustrated in Figure 4. Martin Gardner famously explored this problem in his long-running *Scientific American* column [5].

As with the original formulation of the pursuit problem, the intrinsic perspective is fertile ground for young researchers, since each choice of flight angle $\alpha(t)$ yields a new and potentially interesting system. One example, where the target chooses a

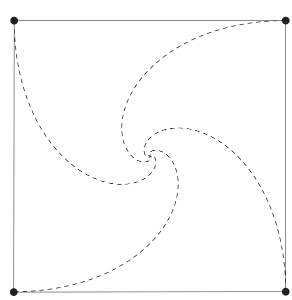

Figure 4. Symmetric pursuit with four bugs.

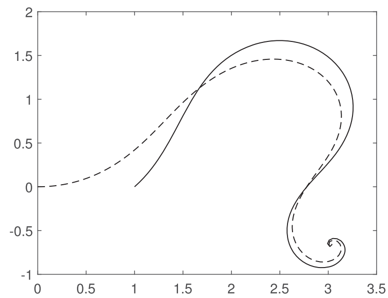

Figure 5. Serpentine flight.

more serpentine route away from its pursuer (represented by $\alpha(t) = \frac{\pi}{4}\cos t$) is shown in Figure 5.

As lovely as all of this is, the real power of the intrinsic perspective comes into play when the pursuit problem itself is viewed from the target's point of view.

3. THE WILD GOOSE CHASE PROBLEM.

Consider now a wily target with motivations beyond simple flight. Perhaps it wants to lead its pursuer into a trap, or away from some more valuable target. What is the subset of points $\boldsymbol{q} \in \mathbb{R}^n$ to which the pursuer can be led? What useful strategies are available to it?

Since the leader (now more naturally interpreted as a goose than a merchant) may view the entire game as being played in the plane defined by $\boldsymbol{m}(0)$, $\boldsymbol{p}(0)$, and $\boldsymbol{q}$, it suffices to consider the case $n = 2$. Moreover, it is natural to posit that the pursuer (which we now imagine to be a farmer) should never be allowed to actually capture its prey, due to both real-world considerations and the mathematical fact that once $\boldsymbol{m}(t) = \boldsymbol{p}(t)$, any pursuit path can be trivially enforced. Finally, we will restrict our attention here to the unit-speed case, $|\boldsymbol{p}'(t)| = |\boldsymbol{m}'(t)| = 1$; readers are encouraged to consider the situation in which one player or the other has a speed advantage.

An essential question immediately arises. Does the goose care how close the farmer ultimately gets, or will any nonzero value suffice? In any practical application, there is bound to exist some minimal safe distance $R_m > 0$ beyond which the pursuer must not be permitted to approach, so we begin with this restriction.

The following lemma does the heavy lifting.

Lemma 1. *For any $\epsilon > 0$, there exists a flight path $\boldsymbol{m}(t)$ for which $\Delta R := R_0 - R_\infty < \epsilon$ but for which the turning of the pursuer $\Delta\theta := \int \kappa(t)\,dt$ is unbounded. One pursuit curve with such properties is given by*

$$\kappa(t) = \frac{c}{1+t},$$

where $c < \frac{\sqrt{\epsilon}}{R_0}$.

In other words, the farmer can be forced to run along a logarithmic spiral, turning indefinitely while closing only an arbitrarily small distance on its target.

Proof. Three things must be proved: that $\Delta\theta$ is in fact unbounded, that the distance closed by the pursuer can be controlled, and that there actually exists a unit-speed flight path giving rise to the pursuit curve described. The first is trivial, since

$$\Delta\theta = \int_0^L \kappa(t)\,dt = \int_0^L \frac{c}{1+t}\,dt \to \infty,$$

as $L \to \infty$, while the claim about ΔR requires only simple approximations. Combining the intrinsic pursuit equations (2) yields

$$\frac{dR}{dt} = -1 \pm \sqrt{1 - R^2\kappa^2} < 0.$$

The positive root is appropriate here, as it is in any case where the target does not charge toward its pursuer. Taking absolute values for convenience and noting that $R \le R_0$ by the monotonicity of R, we then calculate

$$\begin{aligned}
\Delta R &= \int_0^L \left[1 - \sqrt{1 - R^2\kappa^2}\right] dt \\
&< \int_0^\infty \left[1 - \sqrt{1 - \frac{R^2\epsilon}{R_0^2(1+t)^2}}\right] dt \\
&\le \int_0^\infty \left[1 - \sqrt{1 - \frac{\epsilon}{(1+t)^2}}\right] dt \\
&= \int_0^\infty \frac{\epsilon/(1+t)^2}{1 + \sqrt{1 - \epsilon/(1+t)^2}}\,dt \\
&< \int_0^\infty \frac{\epsilon}{(1+t)^2}\,dt = \epsilon.
\end{aligned}$$

Finally, showing that there exists a flight path with the specified $\kappa(t)$ necessitates showing that there is a solution to the following IVP for $\kappa(t) = \frac{c}{1+t}$:

$$\frac{dR}{dt} = -1 + \sqrt{1 - R^2\kappa^2}, \qquad R(0) = R_0. \tag{3}$$

To accomplish this, let $L > 0$ be arbitrary and consider the transformation T on the set $M = \{R \in C^1[0, L] : \|R\|_\infty \le R_0, \|R'\|_\infty \le 1\}$ defined by

$$(TR)(t) = R_0 - \int_0^t \left[1 - \sqrt{1 - R^2\kappa^2}\right] d\tau. \tag{4}$$

The transformation T is continuous, while the set M is both convex and compact by the Arzela–Ascoli theorem, since it consists of a uniformly bounded family of equicontinuous functions in $C^1[0, L]$. Schauder's theorem applies.

Schauder's Theorem [7]. *Let M be a compact convex subset of a Banach space, and T a continuous map of M into itself. Then T has a fixed point.*

In the present case, we need only show that T maps M into itself, that is, that $\|TR\|_\infty \le R_0$ and $\|(TR)'\|_\infty \le 1$. Here, the specific function $\kappa(t)$ must finally be considered. For $\kappa(t) = \frac{c}{1+t}$, $c < \frac{\sqrt{\epsilon}}{R_0}$, we have for small ϵ that

$$1 - R^2\kappa^2 > 1 - \frac{R^2\epsilon}{R_0^2(1+t)^2} \ge 1 - \frac{\epsilon}{(1+t)^2} > 0.$$

Since the inequality $1 - \kappa^2 R^2 \ge 0$ is everywhere satisfied, both $\|TR\|_\infty \le R_0$ and $\|(TR)'\|_\infty \le 1$ hold by equation (4). We conclude that there exists $R(t) \in C^1[0, L]$ satisfying

$$R(t) = R_0 - \int_0^t \left[1 - \sqrt{1 - R^2\kappa^2}\right] d\tau.$$

This holds for any $L > 0$, guaranteeing the existence of a solution to system (3) on $[0, \infty)$ and thus completing the proof of the lemma. ■

Lemma 1 allows the goose to bring the farmer to any desired bearing by guiding it along a logarithmic spiral. With this tool in hand, the pursuer can easily be maneuvered anywhere in the plane.

Theorem 1. *Let $\boldsymbol{p}(0) = (0, 0)$ and $\boldsymbol{m}(0) = (R_0, 0)$ be the starting configuration of the unit-speed wild goose chase problem, and let $\boldsymbol{q}$ be the target ending point. Then, for any $\epsilon > 0$, there exists a flight path $\boldsymbol{m}(t)$ that brings the pursuer to $\boldsymbol{q}$ while allowing it to close no more than distance $\Delta R = \epsilon$.*

Proof. Assume for convenience that $\boldsymbol{q}$ lies in the upper half-plane. Let $\boldsymbol{\gamma_c}(t)$ be the first half-turn of the logarithmic spiral provided by Lemma 1, and let $\boldsymbol{m_c}(t)$ be the flight path that forces it. If $\boldsymbol{q}$ lies on $\boldsymbol{\gamma_c}$, then $\boldsymbol{m_c}(t)$ satisfies the conclusion of the theorem. If $\boldsymbol{q}$ lies inside the closed region formed by $\boldsymbol{\gamma_c}$ and the negative x-axis, then the goose may begin by fleeing an appropriate distance d to the right, as pictured in Figure 6, before leading its pursuer along an identical logarithmic spiral. Finally, if $\boldsymbol{q}$ lies outside $\boldsymbol{\gamma_c}$, then it will lie on $\boldsymbol{\gamma_{\tilde{c}}}$ for some smaller $\tilde{c} < c$. The goose may lead its mark along this path instead simply by following a less curved course. ■

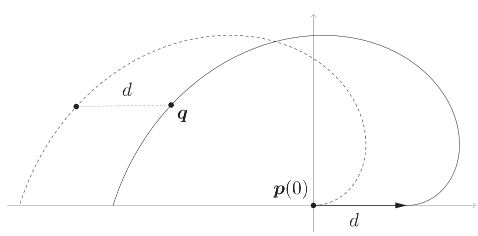

Figure 6. The goose wins.

Theorem 1 says, in essence, that the pursuer can never win. Be it a farmer, a beastly predator, or a dread pirate, it is powerless to prevent its prey from leading it anywhere in space it might desire, and without even the possibility of narrowing the gap in any meaningful way.

The pursuer's only consolation, it seems, lies in the fact that the strategy outlined in Theorem 1 is generally quite inefficient. Logarithmic spirals are slow, gradual arcs, after all, and it may take quite a long time for the game to conclude.

This turns out to be small solace, as other strategies may well be open to the prey.

4. SAFE DISTANCE AND THE KAMIKAZE GOOSE. It is entirely possible for the goose to improve its performance for certain minimal safe distances R_m, and we close by considering one such possibility.

Imagine a goose that elects to ignore the farmer entirely and simply to charge directly at the target point. Against a strategy of pure pursuit, which makes no effort to predict the future course of the prey, this kamikaze approach turns out to be surprisingly effective, and the goose is able to reach its goal in all but the most unfavorable circumstances.

The situation is illustrated in Figure 7. Here, the goose begins at $(R_0, 0)$, as usual, and moves at unit speed in a fixed direction determined by the angle $\alpha \in [0, 2\pi)$. Note that α now represents an absolute bearing, not a relative one.

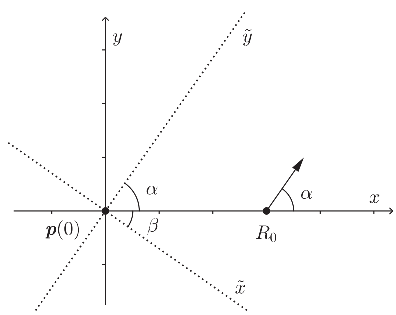

Figure 7. Flight along a fixed bearing.

The pursuit curve can be explicitly calculated, though the details are excruciating (see [**3**] or [**4**] for the full calculation). The key observation is that by considering a rotated frame of reference (also shown in Figure 7), it is possible to find two different expressions involving the parameter t:

$$\frac{d\tilde{y}}{d\tilde{x}} = \frac{R_0 \sin\beta + t - \tilde{y}}{R_0 \cos\beta - \tilde{x}}; \qquad \text{(the pursuit equation)}$$

$$t = \int_0^{\tilde{x}} \sqrt{1 + \left(\frac{d\tilde{y}}{d\xi}\right)^2}\, d\xi. \qquad \text{(the arc length equation)} \tag{5}$$

The latter explicitly makes use of the fact that $\boldsymbol{p}$ is a unit-speed curve, so the arc length function $s(t)$ is just t itself. The interested student can solve the first equation for t, equate this with the second, differentiate, solve for $\frac{d\tilde{y}}{d\tilde{x}}$ and integrate. Writing $\tilde{x}_0 = R_0 \cos\beta$ and restricting $\beta \neq -\frac{\pi}{2}, \frac{\pi}{2}$ (these are the trivial cases where the goose runs directly toward and away from the pursuer, respectively), the results are

$$\frac{d\tilde{y}}{d\tilde{x}} = \frac{1}{2}\left[\frac{\tilde{x}_0(\tan\beta + \sec\beta)}{\tilde{x}_0 - \tilde{x}} - \frac{\tilde{x}_0 - \tilde{x}}{\tilde{x}_0(\tan\beta + \sec\beta)}\right]; \tag{6}$$

$$\tilde{y} = \frac{\tilde{x}_0}{2}(\tan\beta + \sec\beta)\ln\left(\frac{x_0}{x_0 - \tilde{x}}\right) - \frac{2\tilde{x}_0\tilde{x} - \tilde{x}^2}{4\tilde{x}_0(\tan\beta + \sec\beta)}. \tag{7}$$

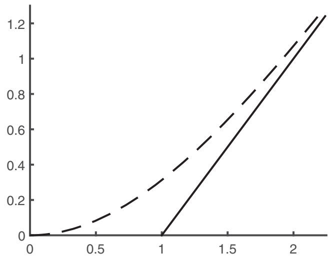

Figure 8. $\alpha = \pi/4$.

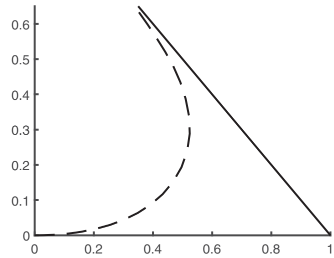

Figure 9. $\alpha = 3\pi/4$.

Figures 8 and 9 show plots of the resulting paths for $\alpha = \pi/4$ and $\alpha = 3\pi/4$, repectively, when $R_0 = 1$. Also note that equation (7) reduces to the solution to Bouguer's problem when $\beta = 0$, as one would hope. In every case, the path of the farmer is asymptotic to that of the goose.

The kamikaze approach may be simple and efficient, but it has the disadvantage of potentially allowing the pursuer to come very close to its prey. In any sort of real-world situation, this is likely to be dangerous. (Just ask the goose as the farmer extends an arm to grab it by the neck.) The distance between pursuer and prey can be explored directly with the following thematic observation:

$$\begin{aligned} R^2 &= (R_0 \cos\beta - \tilde{x})^2 + (R_0 \sin\beta + t - \tilde{y})^2 \\ &= (R_0 \cos\beta - \tilde{x})^2 \left[1 + \left(\frac{d\tilde{y}}{d\tilde{x}} \right)^2 \right]. \end{aligned}$$

Substituting the representation of $\frac{d\tilde{y}}{d\tilde{x}}$ given in equation (6) and simplifying yields

$$R = \frac{1}{2} \left[\tilde{x}_0(\tan\beta + \sec\beta) + \frac{(\tilde{x}_0 - \tilde{x})^2}{\tilde{x}_0(\tan\beta + \sec\beta)} \right]. \tag{8}$$

At the end of this mess lies an elegant conclusion: as $t \to \infty$, we have $\tilde{x} \to \tilde{x}_0$, and hence $R \to \frac{1}{2}\tilde{x}_0(\tan\beta + \sec\beta) = \frac{1}{2}R_0(1 + \sin\beta) := R_\infty$. This is the limiting distance between the players when the flight path is linear; the pursuer can never hope to come closer. Note, in particular, that capture does not occur unless $\beta = -\frac{\pi}{2}$, that is, the goose charges directly into the farmer's clutches.

Given that the kamikaze approach almost never results in capture, it is important to determine the circumstances under which it will be a preferable strategy to the one developed in Section 3. After all, if the goose can reach its target while maintaining the required minimum distance R_m simply by running in a straight line, then it shouldn't bother with those long logarithmic spirals.

Let d be the distance from the starting point $(R_0, 0)$ to the target point $\boldsymbol{q}$, and let β be the corresponding direction angle. Then $\boldsymbol{q}$ is reachable by the kamikaze method if R (as given in equation (8)) is greater than R_m when the players have each traveled distance d. Since $\boldsymbol{p}$ is unit-speed, these quantites are all linked by the terminal value $\tilde{x}(d)$. Direct computation is possible.

Continuing to write $R_\infty = \frac{1}{2}R_0(1 + \sin\beta)$, we substitute equation (6) into equation (5) and simplify.

$$d = \frac{1}{2}\int_0^{\tilde{x}} \left(\frac{2R_\infty}{\tilde{x}_0 - \xi} + \frac{\tilde{x}_0 - \xi}{2R_\infty} \right) d\xi$$
$$= R_\infty \ln\left(\frac{\tilde{x}_0}{\tilde{x}_0 - \tilde{x}} \right) + \frac{\tilde{x}_0^2 - (\tilde{x}_0 - \tilde{x})^2}{8R_\infty}.$$

Writing equation (8) as $(\tilde{x}_0 - \tilde{x}) = 2\sqrt{R_\infty(R - R_\infty)}$ then gives us an upper bound on d in terms of R_m and β:

$$d_{max} = R_\infty \ln\left(\frac{\tilde{x}_0}{2\sqrt{R_\infty(R_m - R_\infty)}} \right) + \frac{\tilde{x}_0^2 - 4R_\infty(R_m - R_\infty)}{8R_\infty}$$
$$= R_\infty \ln\left(\frac{R_0 \cos\beta}{2\sqrt{R_\infty(R_m - R_\infty)}} \right) + \frac{1}{2}(R_0 - R_m). \tag{9}$$

Equation (9) is rather inscrutable, so a computer plot is appropriate; see Figure 10.

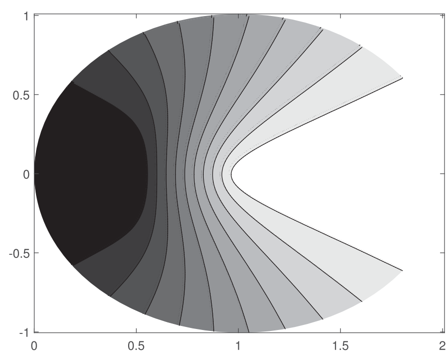

Figure 10. Achieving $R = R_m$.

Here we see the locations at which a goose starting at $(1, 0)$ will be captured by a farmer with arms of length R_m when fleeing in various directions β. For a given target point $\boldsymbol{q}$, the goose will win if $\boldsymbol{q}$ lies to the right of the corresponding isoline of R_m, which in this figure increment in steps of 0.1. These curves thus show where the kamikaze approach becomes unworkable and another strategy, such as the logarithmic spiral method, is necessary.

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